Sub Topic : Spin Angular Momentum

1. What is spin angular momentum →

History : $\bar{e}$'s spin (spin angular momentum) was discovered through Stern - Gerlanch Experiment.

- Silver atoms (Ag) were used.
- Ag has 47 $\bar{e}$'s. 46 of them forms a spherically symmetric charge distribution and 47th $\bar{e}$ occupy 5s orbital.
- In ground state, angular momentum ($l=0$) must be zero for Ag as valence $\bar{e}$ is in 5s-orbital.
- In SG Experiment, non-homogenous M.F is applied along z-axis.

According to schrodinger model.

→ Beam of Ag atoms should go undeflected and we must get Just one spot on screen. if Ag is in ground state (5s) $l=0$

→ If Ag is in excited state (5p) $l=1$. Beam will split into $2l+1 = 3$ beams and we get 3 spots on screen.

- Experimentally, beam splits into 2 components and we observe 2 spots on screen for $l=0$. Where no spliting was expected.
- Conclusion:

→ $\bar{e}$ posses an intrinsic angular momentum which is called spin angular momentum.

→ It is analog to earth rotating around its own axis.

→ In real, $\bar{e}$ is point particle and it has no axis or rotation. $\bar{e}$ is born with this intrinsic spin.

Important:

(1) Spin angular momentum does not depend upon its spatial DOF.

(2) Spin is an intrinsic DOF.

(3) Spin is purely QM concept which has no classical analog.

(4) Unlike orbital angular momentum (L), spin can't be described by differential operator.

Magnetic Moment (μ_s) :-

- Magnetic moment of any body having angular momentum is

$$\vec{\mu}_s = -\frac{e\vec{S}}{2m} \rightarrow \text{classical prediction}$$

This would be true if $\bar{e}$ is assumed to charged sphere rotating around its own axis.

- In real, $\vec{\mu}_s = -2 \cdot \frac{e\vec{S}}{2m} = -g_s \frac{e}{2m}\vec{S}$

($g_s = 2$ lande factor)

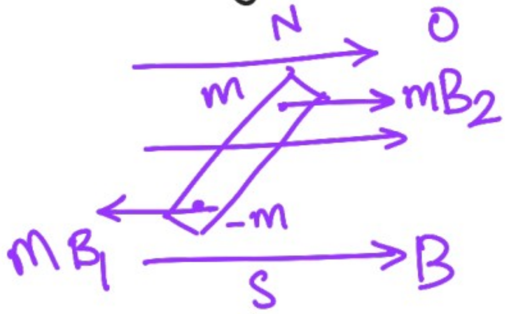
It is double the classical prediction.

External M·F →

qE

- IF $\bar{e}$ with magnetic moment μ_s is placed in external M·F (non-homogenous) It will experience force.

$$\boxed{\vec{F} = \vec{\nabla}(\vec{\mu_s} \cdot \vec{B})}$$



WHY?

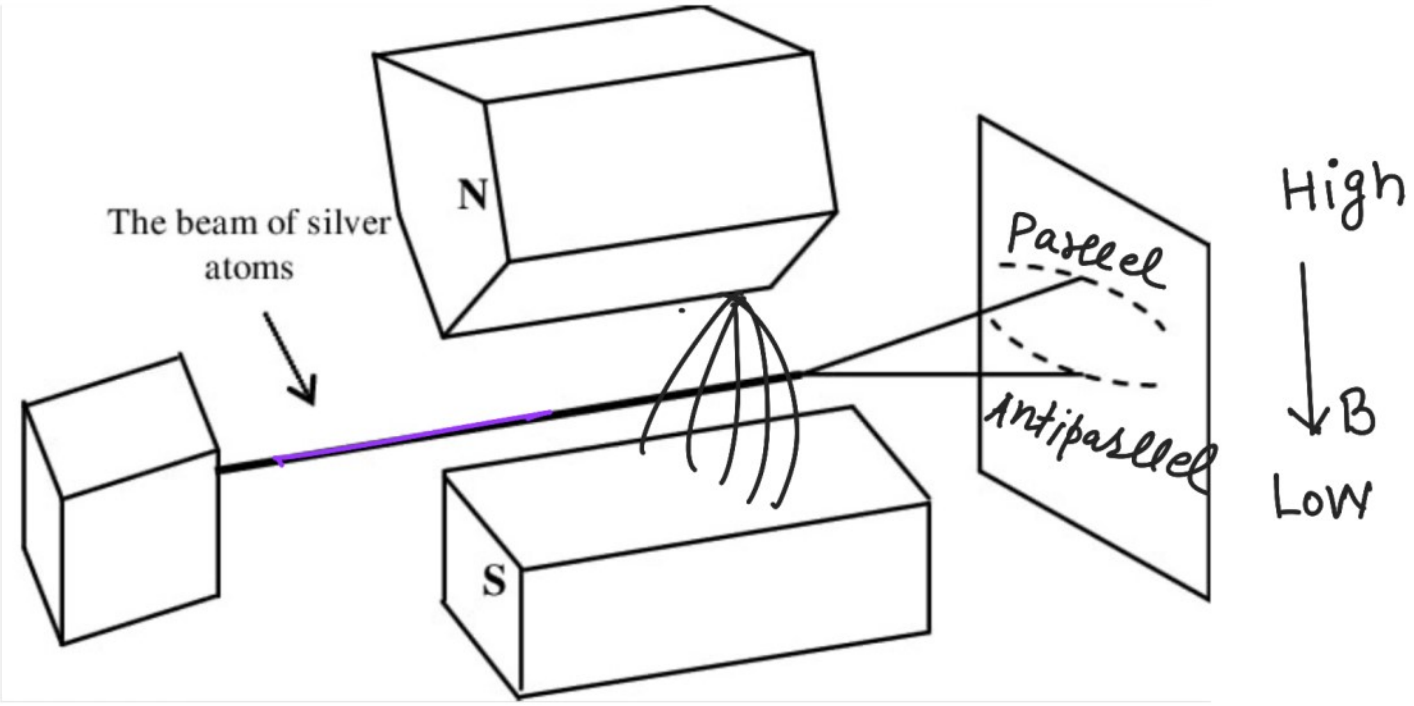
$$\vec{F} = -\vec{\nabla}U = \vec{\nabla}(\vec{\mu} \cdot \vec{B}) \quad (U = -\vec{\mu} \cdot \vec{B})$$

- Direction and magnitude of force depends upon relative orientation of field and dipole.

- If μ_s is in parallel to M·F : $\bar{e}$ will move in direction in which field increase.

- If μ_s is antiparallel to M·F : $\bar{e}$ will move in direction in which field decrease.

(I dare you to explain above 2 points)



Flat side magnet (low M·F) field lines far
Curved " " (High M·F) " " close

Orientation with M·F

- spin of any particle is described by spin Q·N (S).
- $S = 1/2$ for $\bar{e}, p^+, n^0$
- Spin can have $2S+1$ possible orientation w·r·t external M·F.
- Each orientation is represented by spin magnetic Q·N (m_s)

$$m_s = -S \text{ to } S \quad \text{in steps of } 1$$

- for $\bar{e}$, $S = 1/2$ $\quad m_s = -1/2 \cdot 1/2$

$\therefore$ $\bar{e}$ beam splits into 2 beams.

$m_s = 1/2$ — Spin parellel to M·F

$m_s = -1/2$ — antiparellel.

- Spin $S = 3/2$ particle beam will split into $2S+1 = 4$ beams $\left(m_s = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$

2. Formalism of Spin →

General Vector :

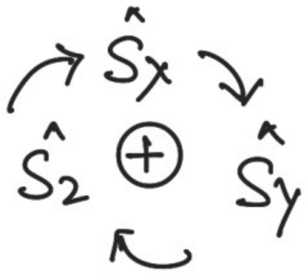
$$\vec{S} = \hat{S}_x \hat{i} + \hat{S}_y \hat{j} + \hat{S}_z \hat{k}$$

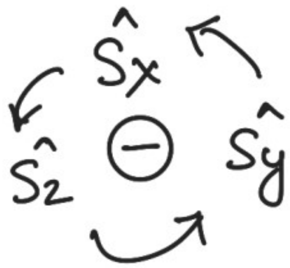
Commutator :

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z \qquad [\hat{S}_y, \hat{S}_x] = -i\hbar \hat{S}_z$$

$$[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x \qquad [\hat{S}_z, \hat{S}_y] = -i\hbar \hat{S}_x$$

$$[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y \qquad [\hat{S}_x, \hat{S}_z] = -i\hbar \hat{S}_y$$





$$[\hat{S}_x, \hat{S}_x] = [\hat{S}_y, \hat{S}_y] = [\hat{S}_z, \hat{S}_z] = 0$$

$\hat{S}_z$ and $\hat{S}^2$:

- $\hat{S}_z$ and $\hat{S}^2$ commute.

Significance

(1) $[\hat{S}_z, \hat{S}^2] = 0$

(2) Both operators have different

eigenvalues but same eigenvectors.

(3) Both can be measured simultaneously with 100% precesion.

(4) Measurement of one operator don't disturb outcome of other.

(5) In general eigenvector of these operators are written as $|s, m_s\rangle$. It describes the state.

Eigenvalue equations

$$\hat{A}\,|s, m_s\rangle = \lambda\,|s, m_s\rangle$$

operator

eigenvector (State)

eigenvalue (measurement result)

eigenvector (State)

$$\hat{S}^2 |s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$$

$$\hat{S}_z |s, m_s\rangle = m_s\hbar\, |s, m_s\rangle$$

* Eigenvalue for $\hat{S}^2$ is $s(s+1)\hbar^2$. If you measure $\hat{S}^2$ for system in state $|s, m_s\rangle$

you will get $s(s+1)\hbar^2$.

* IIIly eigenvalues of $\hat{S}_z$ is $m_s\hbar$.

Laddar operators :

$$\boxed{\begin{aligned}\hat{S}_+ &= \hat{S}_x + i\hat{S}_y \\ \hat{S}_- &= S_x - i\hat{S}_y\end{aligned}}$$

$\hat{S}_+ \rightarrow$ Increase ms by +1

$\hat{S}_- \rightarrow$ decrease ms by −1

$$\boxed{\begin{aligned}\hat{S}_+|s,m_s\rangle &= \sqrt{s(s+1)-m_s(m_s+1)}\,\hbar\,|s,m_s+1\rangle \\ \hat{S}_-|s,m_s\rangle &= \sqrt{s(s+1)-m_s(m_s-1)}\,\hbar\,|s,m_s-1\rangle\end{aligned}}$$

Square operators :

- Average values : $\langle \hat{S}_x^2\rangle = \langle \hat{S}_y^2\rangle = ?$

$$\boxed{\begin{aligned}\langle \hat{S}_z^2\rangle &= m_s^2\hbar^2 \\ \langle \hat{S}^2\rangle &= s(s+1)\hbar^2\end{aligned}}$$

- $\langle \hat{S}^2\rangle = \langle \hat{S}_x^2\rangle + \langle \hat{S}_y^2\rangle + \langle \hat{S}_z^2\rangle$

$$s(s+1)\hbar^2 = 2\langle \hat{S}_x^2\rangle + m_s^2\hbar^2$$

$$\boxed{\langle \hat{S}_x^2 \rangle = \langle \hat{S}_y^2 \rangle = \left(S(S+1) - m_s^2\right)\frac{\hbar^2}{2}}$$

Average of any operator :

- Average of operator $\hat{A}$ for system in state $|S, m_s\rangle$ is

$$\boxed{\langle \hat{A} \rangle = \langle S, m_s | \hat{A} | S, m_s \rangle}$$

- What does it mean?

System is in state $|S, m_s\rangle$. We measure physical quantity $\hat{A}$ for this state. We may get different value on each measurement. Average of physical quantity $\hat{A}$ is $\langle \hat{A} \rangle$.

Orthonormal states :

- Spin states form orthonormal basis

Orthogonal to each other

$$\langle S', m_s' | S, m_s \rangle = 0 \quad \begin{pmatrix} S' \neq S \\ m_s' \neq m_s \end{pmatrix}$$

Normalized

$$\langle S, m_s | S, m_s \rangle = 1$$

- They form complete basis.

$$\sum_{m_s=-s}^{s} |s, m_s\rangle\langle s, m_s| = I$$

Question: Jest 2022

Consider a spin-1 system whose $\hat{S}_z$ eigenstates are given by $|-1\rangle, |0\rangle, |+1\rangle$ corresponding to the eigenvalues $-\hbar, 0, \hbar$. The normalized general state $|\psi\rangle$ of the system can be expressed as

$$|\psi\rangle = c_{-1}|-1\rangle + c_0|0\rangle + c_{+1}|+1\rangle,$$

and c_{-1}, c_0, c_{+1} are complex numbers. Subjected to the condition $\langle\psi|\hat{S}_z|\psi\rangle = 0$, which of the following statements is true?

(a) $|c_{+1}|^2 + 2|c_0|^2 = 1$
(b) $|c_{-1}|^2 + 2|c_0|^2 = 1$
(c) $2|c_{-1}|^2 + |c_{+1}|^2 = 1$
(d) $2|c_{-1}|^2 + |c_0|^2 = 1$

Solution:

$S = 1 \quad m_s = -1, 0, 1 \equiv |-1\rangle, |0\rangle, |1\rangle$

- $\hat{S}_z|-1\rangle = -\hbar|-1\rangle$

  $\hat{S}_z|0\rangle = 0$

  $\hat{S}_z|1\rangle = \hbar|1\rangle$

- $|\psi\rangle = C_{-1}|-1\rangle + C_0|0\rangle + C_{+1}|1\rangle$

- $\langle\psi|\hat{S}_z|\psi\rangle = ?$

- $\hat{S}_z|\psi\rangle = \hat{S}_z\left(C_{-1}|-1\rangle + C_0|0\rangle + C_1|1\rangle\right)$

$$= C_{-1}(-\hbar|-1\rangle) + 0 + C_1(\hbar|1\rangle)$$

$$= \hbar\left(-C_{-1}|-1\rangle + C_1|1\rangle\right)$$

- $\langle\psi|\hat{S}_z|\psi\rangle = 0$

$$\Rightarrow \left(C_{-1}^*\langle -1| + C_0^*\langle 0| + C_1^*\langle 1|\right)\left(-C_{-1}|-1\rangle + C_1|1\rangle\right)\hbar = 0$$

$$\Rightarrow -C_{-1}^*C_{-1} + 0 + C_1^*C_1 = 0$$

(only simmilar terms give non zero)

$$\Rightarrow \boxed{|C_1|^2 - |C_{-1}|^2 = 0} \quad \text{— (1)}$$

From Normalization of Wavefunction

$$\langle\psi|\psi\rangle = 1$$

$$\left(C_{-1}^*\langle -1| + C_0^*\langle 0| + C_1^*\langle 1|\right)\left(C_{-1}|-1\rangle + C_0|0\rangle + C_1|1\rangle\right) = 1$$

$$\boxed{|C_{-1}|^2 + |C_0|^2 + |C_1^2| = 1} \quad \text{— (2)}$$

Subtract ① and ②

$$\boxed{2|c_{-1}|^2 + |c_0|^2 = 1}$$ (d)

Question: Jest 2023

Consider a spin-1/2 particle in the quantum state $|\psi(\beta,\alpha)\rangle = \cos\left(\frac{\beta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|\downarrow\rangle$ where $0 \le \beta \le \pi$ and $0 \le \alpha \le 2\pi$. For which values of (δ,γ) is the state $|\psi(\delta,\gamma)\rangle$ orthogonal to $|\psi(\beta,\alpha)\rangle$?

(a) $(\pi+\beta, \pi-\alpha)$ (b) $(\pi-\beta, \pi-\alpha)$ (c) $(\pi+\beta, \pi+\alpha)$ (d) $(\pi-\beta, \pi+\alpha)$

Solution:

$$|\psi(\beta,\alpha)\rangle = \cos(\beta/2)|\uparrow\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|\downarrow\rangle$$

$$0 \le \beta \le \pi \quad \text{and} \quad 0 \le \alpha \le 2\pi$$

$|\psi(\delta,\gamma)\rangle$ orthogonal to $|\psi(\beta,\alpha)\rangle$

$$\langle\psi(\delta,\gamma)|\psi(\beta,\alpha)\rangle = 0$$

$$\left(\cos^*(\delta/2)\langle\uparrow| + \sin^*(\delta/2)e^{-i\gamma}\langle\downarrow|\right)\left(\cos\left(\frac{\beta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|\downarrow\rangle\right) = 0$$

$$\cos^*(\delta/2) = \cos(\delta/2)$$
$$\sin^*(\delta/2) = \sin(\delta/2)$$
(Prove it!)

$$\cos\left(\frac{\delta}{2}\right)\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\delta}{2}\right)\sin\left(\frac{\beta}{2}\right)e^{i(\alpha-\gamma)} = 0$$

Assume, $e^{i(\alpha-\gamma)} = -1 \quad (= e^{i0}, e^{i(\pi)}, \ldots)$

$\alpha - \gamma = 0, \pi, \ldots \quad \Rightarrow \quad \boxed{\gamma = \alpha - \pi}$

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$\therefore \frac{1}{2}\left(\cos\left(\frac{\delta+\beta}{2}\right) + \cos\left(\frac{\delta-\beta}{2}\right) - \cos\left(\frac{\delta+\beta}{2}\right) + \cos\left(\frac{\delta-\beta}{2}\right)\right) = 0$$

$$\therefore \cos\left(\frac{\delta-\beta}{2}\right) = 0$$

$$\frac{\delta-\beta}{2} = \frac{\pi}{2} \Rightarrow \delta - \beta = \pi$$

$$\boxed{\delta = \beta + \pi}$$

$\boxed{\gamma = \pi - \alpha}$ if we do $\langle \psi(\beta,\alpha) | \psi(\delta,\gamma) \rangle = 0$

(Option a)

3. Matrix Representation →

$$\boxed{S = 1/2 \text{ particle}}$$

- $S = 1/2 \qquad m_s = \pm 1/2$
- Possible states $|s, m_s\rangle = |1/2, 1/2\rangle, \; |1/2, -1/2\rangle$

$|1/2, 1/2\rangle \hookrightarrow |\uparrow\rangle \qquad |1/2, -1/2\rangle \hookrightarrow |\downarrow\rangle$

$\hat{S}^2$ - Matrix

$$\hat{S}^2 = \begin{pmatrix} \langle\uparrow|\hat{S}^2|\uparrow\rangle & \langle\uparrow|\hat{S}^2|\downarrow\rangle \\ \langle\downarrow|\hat{S}^2|\uparrow\rangle & \langle\downarrow|\hat{S}^2|\downarrow\rangle \end{pmatrix}$$

$$\hat{S}^2|\uparrow\rangle = \hbar^2 s(s+1)|\uparrow\rangle = \frac{3}{4}\hbar^2|\uparrow\rangle$$

$$\hat{S}^2|\downarrow\rangle = \frac{3}{4}\hbar^2|\downarrow\rangle$$

$$\hat{S}^2 = \begin{pmatrix} \frac{3\hbar^2}{4}\langle\uparrow|\uparrow\rangle & \frac{3\hbar^2}{4}\langle\uparrow|\downarrow\rangle \\ \frac{3\hbar^2}{4}\langle\downarrow|\uparrow\rangle & \frac{3\hbar^2}{4}\langle\downarrow|\downarrow\rangle \end{pmatrix}$$

$$\boxed{\hat{S}^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$\hat{S}_z$ - Matrix :

$$\hat{S}_z = \begin{pmatrix} \langle \uparrow | S_z | \uparrow \rangle & \langle \uparrow | S_z | \downarrow \rangle \\ \langle \downarrow | S_z | \uparrow \rangle & \langle \downarrow | S_z | \downarrow \rangle \end{pmatrix}$$

$$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \quad \text{and} \quad \hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$$

$$\hat{S}_z = \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$$

$$\boxed{\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$\hat{S}_+$ Matrix :

$$\hat{S}_+ = \begin{pmatrix} \langle \uparrow | \hat{S}_+ | \uparrow \rangle & \langle \uparrow | \hat{S}_+ | \downarrow \rangle \\ \langle \downarrow | \hat{S}_+ | \uparrow \rangle & \langle \downarrow | \hat{S}_+ | \downarrow \rangle \end{pmatrix}$$

$$\hat{S}_+ |\uparrow\rangle = \hat{S}_+ \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \sqrt{S(S+1) - m_s(m_s+1)}\,\hbar = 0$$

$$\hat{S}_+ |\downarrow\rangle = \hat{S}_- \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \sqrt{\frac{3}{4} + \frac{1}{4}}\,\hbar \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \hbar |\uparrow\rangle$$

$$\hat{S}_+ = \begin{pmatrix} 0 & \hbar \langle \uparrow | \uparrow \rangle \\ 0 & \hbar \langle \downarrow | \uparrow \rangle \end{pmatrix} = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix}$$

$$\boxed{\hat{S}_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}$$

$\hat{S}_-$ Matrix :

$$\hat{S}_- = \begin{pmatrix} \langle\uparrow|\hat{S}_-|\uparrow\rangle & \langle\uparrow|\hat{S}_-|\downarrow\rangle \\ \langle\downarrow|\hat{S}_-|\uparrow\rangle & \langle\downarrow|\hat{S}_-|\downarrow\rangle \end{pmatrix}$$

$$\hat{S}_-|\uparrow\rangle = \hat{S}_-\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \sqrt{S(S+1)-m_s(m_s-1)}\,\hbar\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

$$= \sqrt{\frac{3}{4}+\frac{1}{4}}\,\hbar\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

$$\hat{S}_-|\uparrow\rangle = \hbar|\downarrow\rangle$$

$$\hat{S}_-|\downarrow\rangle = 0$$

$$\hat{S}_- = \begin{pmatrix} \langle\uparrow|\downarrow\rangle\hbar & 0 \\ \langle\downarrow|\downarrow\rangle\hbar & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix}$$

$$\boxed{\hat{S}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$$

$\hat{S}_x$ Matrix :

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{i}{2}(\hat{S}_- - \hat{S}_+) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli- Matrices :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties

(1) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$ (involuntary matrices)

(2)

Commutators

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$

$$[\sigma_y, \sigma_z] = 2i\sigma_x$$

$$[\sigma_z, \sigma_x] = 2i\sigma_y$$

Anticommutators

$$\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$$

$$\{\sigma_x, \sigma_y\} = 0$$

$$\{\sigma_y, \sigma_z\} = 0$$

$$\{\sigma_z, \sigma_x\} = 0$$

(3) Determinant $= -1$

(4) Trace $= 0$

(5) Eigenvalues $= \pm 1$

(6) Eigenvectors

$$\psi_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \psi_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\psi_{y+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \psi_{y-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\psi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Spin Matrices

$$S_x = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\sigma_y \qquad S_z = \frac{\hbar}{2}\sigma_z$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

Formulae to help problem Solving :

① $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = (\vec{A}\cdot\vec{B})\hat{I} + i\,\vec{\sigma}\cdot(\vec{A}\times\vec{B})$

② $\sigma_x\sigma_y\sigma_z = i\hat{I}$

③ $e^{i\alpha\sigma_j} = \hat{I}\cos\alpha + i\,\sigma_j\sin\alpha \qquad (j = x, y, z)$

commutators with other Quantities →

- Spin does not depend on spatial DOF, so spin commute with all spatial operators

$$[\hat{S}_j, \hat{L}_k] = 0 \qquad [\hat{S}_j, \hat{R}_k] = 0 \qquad [\hat{S}_j, \hat{P}_k] = 0$$

eg $[S_x, L_z] = 0 \qquad [S_y, z] = 0 \qquad [S_x, P_y] = 0$

Spin operators in terms of pauli matrices

For $s = 1/2$

$$S_x = \frac{\hbar}{2}\sigma_x \qquad S_y = \frac{\hbar}{2}\sigma_y \qquad S_z = \frac{\hbar}{2}\sigma_z$$

eigenvalues

$$\pm\hbar/2 \qquad \pm\hbar/2 \qquad \pm\hbar/2$$

eigenvectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \qquad \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$$

Eigenvectors of $\hat{S}_z$ as basis vectors :

- $\hat{S}_z$ has 2 eigenvectors $|\uparrow\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|\downarrow\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- In general state of system can be written as :

$$\boxed{\psi = \alpha|\uparrow\rangle_z + \beta|\downarrow\rangle_z}$$

Probability to get $|\uparrow\rangle_z$ (Spin up) $= \dfrac{\alpha^2}{\alpha^2+\beta^2}$

" " " $|\downarrow\rangle_z$ (Spin down) $= \dfrac{\beta^2}{\alpha^2+\beta^2}$

- How can we write our state in terms of eigenvectors of $\hat{S}_x$?

$|\uparrow\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad$ $|\downarrow\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

we need to write $|\uparrow\rangle_z$, $|\downarrow\rangle_z$ in terms of $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$

let, $|\uparrow\rangle_z = a|\uparrow\rangle_x + b|\downarrow\rangle_x$

$$a = {}_x\langle\uparrow|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(1 \;\; 1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$b = {}_x\langle\downarrow|\uparrow_z\rangle_z = \frac{1}{\sqrt{2}}(1 \;\; -1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \boxed{|\uparrow\rangle_z = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x + |\downarrow\rangle_x\right)}$$

llly, let $|\downarrow\rangle_z = a|\uparrow\rangle_x + b|\downarrow\rangle_x$

$$a = {}_x\langle\uparrow|\downarrow\rangle_z = \frac{1}{\sqrt{2}}(1 \;\; 1)\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$b = {}_x\langle\downarrow|\downarrow\rangle_z = \frac{1}{\sqrt{2}}(1 \;\; -1)\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{\sqrt{2}}$$

$$\therefore \quad \boxed{|\downarrow\rangle_z = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x + |\downarrow\rangle_x\right)}$$

$$\therefore \psi = \alpha |\uparrow\rangle_z + \beta |\downarrow\rangle_z$$

$$\psi = \frac{\alpha}{\sqrt{2}}\left(|\uparrow\rangle_x + |\downarrow\rangle_x\right) + \frac{\beta}{\sqrt{2}}\left(|\uparrow\rangle_x - |\downarrow\rangle_x\right)$$

$$\boxed{\psi = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|\uparrow\rangle_x + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|\downarrow\rangle_x}$$

Probability to get $|\uparrow\rangle_x$ $= \dfrac{\left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2}{\left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2 + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)^2}$

Probability to get $|\downarrow\rangle_x$ $= \dfrac{\left(\frac{\alpha-\beta}{\sqrt{2}}\right)^2}{\left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2 + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)^2}$

- How can we write our state in terms of eigenvectors of $\hat{S}_y$.

$$|\uparrow\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \qquad |\uparrow\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

let, $|\uparrow\rangle_z = a|\uparrow\rangle_y + b|\downarrow\rangle_y$

$a = {}_y\langle\uparrow|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(1 \ -i)\begin{pmatrix}1\\0\end{pmatrix} = 1/\sqrt{2}$

$b = {}_y\langle\downarrow|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(1 \ \ i)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$

$\therefore \boxed{|\uparrow\rangle_z = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_y + |\downarrow\rangle_y\right)}$

let, $|\downarrow\rangle_z = a|\uparrow\rangle_y + b|\downarrow\rangle_y$

$a = {}_y\langle\uparrow|\downarrow\rangle_z = \frac{1}{\sqrt{2}}(1 \ -i)\begin{pmatrix}0\\1\end{pmatrix} = \frac{-i}{\sqrt{2}}$

$b = {}_y\langle\downarrow|\downarrow\rangle_z = \frac{1}{\sqrt{2}}(1 \ \ i)\begin{pmatrix}0\\1\end{pmatrix} = \frac{i}{\sqrt{2}}$

$\boxed{|\downarrow\rangle_z = \frac{i}{\sqrt{2}}\left(-|\uparrow\rangle_y + |\downarrow\rangle_y\right)}$

$\therefore \ \psi = \alpha|\uparrow\rangle_z + \beta|\downarrow\rangle_z$

$\psi = \frac{\alpha}{\sqrt{2}}\left(|\uparrow\rangle_y + |\downarrow\rangle_y\right) + \frac{\beta i}{\sqrt{2}}\left(-|\uparrow\rangle_y + |\downarrow\rangle_y\right)$

$$\boxed{\Psi = \frac{(\alpha - i\beta)}{\sqrt{2}} |\uparrow\rangle_y + \left(\frac{\alpha + i\beta}{\sqrt{2}}\right) |\downarrow\rangle_y}$$

Question: Spin 1/2 particle is in state

$$\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

a) If you measure S_z, what is probability that you will get $\pm\hbar/2$.

$$\chi = \frac{1+i}{\sqrt{6}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{\sqrt{6}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\chi = \alpha\, |\uparrow\rangle_z + \beta\, |\downarrow\rangle_z \qquad \left(\begin{matrix} \alpha = \dfrac{1+i}{\sqrt{6}} \\ \beta = \dfrac{2}{\sqrt{6}} \end{matrix}\right)$$

$$\alpha^2 + \beta^2 = \left|\frac{1+i}{\sqrt{6}}\right|^2 + \left|\frac{2}{\sqrt{6}}\right|^2 = \frac{2}{6} + \frac{4}{6} = 1$$

$$P(= +\hbar/2) = \left|\frac{1+i}{\sqrt{6}}\right|^2 = \frac{2}{6} = 1/3$$

$$P(= -\hbar/2) = \left|\frac{2}{\sqrt{6}}\right|^2 = \frac{4}{6} = 2/3$$

b) If you measure $\hat{S}_x$. What is the probability of getting $\pm\hbar/2$.

$$P(=\hbar/2) = \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \left|\frac{3+i}{\sqrt{12}}\right|^2 = \frac{10}{12} = \frac{5}{6}$$

$$P(=-\hbar/2) = \left|\frac{\alpha-\beta}{\sqrt{2}}\right|^2 = \left|\frac{-1+i}{\sqrt{12}}\right|^2 = \frac{2}{12} = \frac{1}{6}$$

c) What is expectation value of $\hat{S}_x$:

$$\langle \hat{S}_x \rangle = \langle \psi | \hat{S}_x | \psi \rangle$$

$$= \begin{pmatrix} \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} (1+i)/\sqrt{2} \\ 2/\sqrt{6} \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} \\ (1+i)/\sqrt{6} \end{pmatrix}$$

$$= \frac{\hbar}{2} \left(\frac{2-2i+2+2i}{6} \right) = \frac{\hbar}{2} \frac{4}{6} = \frac{\hbar}{3}$$

Spin Operator in any random direction

- Find eigenvalues and eigenvectors of spin operator $\vec{S}$ of an electron in the direction of unit vector $\vec{n}$ ($\vec{n}$ is arbitary).

- Operator : $\vec{n} \cdot \vec{S}$

- eigenvalue equation :

$$\vec{n} \cdot \vec{S} |\psi\rangle = \frac{\hbar}{2} \lambda |\psi\rangle$$

- $\vec{n}$ in spherical coordinate system

$$\vec{n} = \sin\theta \cos\phi \,\hat{i} + \sin\theta \sin\phi \,\hat{j} + \cos\theta \,\hat{k}$$

- $\vec{S} = S_x \hat{i} + S_y \hat{j} + S_z \hat{k}$

- $\vec{n} \cdot \vec{S} = (S_x \sin\theta \cos\phi) + (S_y \sin\theta \sin\phi) + (S_z \cos\theta)$

$$\vec{n}\cdot\vec{S} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi$$

$$+\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

eigenvalues

$$\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin\theta\, e^{-i\phi} \\ \frac{\hbar}{2}\sin\theta\, e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} = 0$$

$$-\frac{\hbar^2}{4}\left(\cos^2\theta - \lambda^2\right) - \frac{\hbar^2}{4}\sin^2\theta = 0$$

$$\frac{\lambda^2 \hbar^2}{4} - \frac{\hbar^2}{4} = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

eigenvectors :

$$|\lambda\rangle_+ = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$$

$$|\lambda\rangle_- = \begin{pmatrix} -\sin\theta/2 \\ e^{i\phi}\cos\theta/2 \end{pmatrix}$$

Example : Jest 2023

Consider the operator $S \cdot \hat{n}$ with eigenkets $|\pm\rangle_{\hat{n}}$ and eigenvalues $\pm\frac{\hbar}{2}$ where $\hat{n}$ is a unit vector and S is the spin operator. A partially polarized beam of spin-$\frac{1}{2}$ particles contains a 25-75 mixture of two pure ensembles, one with $|+\rangle_{\hat{z}}$ and the other with $|+\rangle_{\hat{x}}$ respectively. What is the ensemble average of

$$\frac{S \cdot \hat{x}}{\hbar}?$$

(a) $\frac{1}{3}$

(b) $\frac{1}{8}$

(c) $\frac{1}{4}$

(d) $\frac{3}{16}$

$\frac{\hat{S}\cdot\hat{x}}{\hbar}$ is simply $\frac{S_x}{\hbar}$

our system is in state

$$\psi = \frac{1}{2}|\uparrow\rangle_z + \frac{\sqrt{3}}{2}|\uparrow\rangle_x$$

$$\psi = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{\sqrt{3}}{2\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{2}+\sqrt{3}\\ \sqrt{3}\end{pmatrix}$$

average of $S_x/\hbar$

$$\frac{\langle\psi|S_x|\psi\rangle}{\hbar} = \frac{\hbar}{2\hbar}\,\frac{1}{8}\begin{pmatrix}\sqrt{2}+\sqrt{3} & \sqrt{3}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{2}+\sqrt{3}\\ \sqrt{3}\end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix}\sqrt{2}+\sqrt{3} & \sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{3}\\ \sqrt{2}+\sqrt{3}\end{pmatrix}$$

$$= \frac{1}{16}\left(2\sqrt{3}(\sqrt{2}+\sqrt{3}\right)$$

$$= \frac{\sqrt{3}(\sqrt{2}+\sqrt{3})}{8}$$

ans not matching